ON SIMILARITY OF TRANSONIC PLANE FLOWS

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Simplified equations of transonic gas motions and criteria of transonic similarity were established by Kármán [1] and, independently, by Fal'kovich for irrotational motions. Ovsiannikov [2] has shown that the equations mentioned can be applied, within the limits of accuracy with regard to satisfying the equations of motion and the boundary conditions, to flows with shock waves. In Ref. [3,4] as well as others, boundary value problems were formulated for approximate equations corresponding to the problem of transonic flow around a wedge-shaped profile for different regimes. However, comparison of the solutions obtained with experimental data exhibits a systematic discrepancy in the coefficients of pressure resistance if the M number of the upstream flow is somewhat different from unity. This fact was noticed by Spreiter [5], who suggested a different method to simplify the equations of motion, and indicated a new similarity rule. Spreiter's method of simplifying the equations is not convincing due to absence of estimates regarding the omitted terms in the equations of motion and in the boundary conditions.

Proceeding by analogy to the method applied by Ovsiannikov [2], the present paper contains a simplification of the basic equations of motion based on the assumption that the M number of the upstream flow is close to unity, the velocities of the flow are only slightly different from the upstream velocity, and that the direction of the velocity vector is only slightly different from the direction of flow away from the body. A similarity rule is established thereby, which yields more accurate results in the study of flow with a passage across the speed of sound (transonic flows).

1. Equations of irrotational gas motions. Let us consider the equations of irrotational motion of plane-parallel flow of an inviscid ideal gas:

$$\frac{\partial \rho v_x}{\partial x} + \frac{\partial \rho v_y}{\partial y} = 0, \qquad \frac{\partial v_x}{\partial y} - \frac{\partial v_y}{\partial x} = 0 \tag{1.1}$$

Here x, y are the coordinates of the flow plane, v_x, v_y are the components of the velocity vector, and ρ is the gas density.

Let

$$\frac{v_x}{v_\infty} = 1 - \varepsilon A U, \qquad \frac{v_y}{v_\infty} = \varepsilon^{3/2} A V$$

$$\frac{y}{l_*} = \varepsilon^{-1/2} Y, \qquad \frac{x}{l_*} = X \tag{1.2}$$

where v_∞ is the upstream velocity, l_* is the characteristic line, ϵ is a small quantity and A is a constant for a given number M_∞ and is smaller than unity.

From Bernoulli's equation it is easily found that

$$\frac{\rho}{\rho_\infty} = \left[1 + \frac{(k-1)}{2} M_\infty^2 \left(1 - \frac{w^2}{v_\infty^2}\right)\right]^{\frac{1}{k-1}} \tag{1.3}$$

$$\frac{p}{p_\alpha} = \left[1 + \frac{(k-1)}{2} M_\infty^2 \left(1 - \frac{w^2}{v_\infty^2}\right)\right]^{\frac{k}{k-1}} \tag{1.4}$$

Designating

$$K = \frac{1 - M_\infty^2}{\varepsilon} \tag{1.5}$$

and considering the quantity K of the order of unity ($1 - M_\infty^2$ of the order of ϵ), we obtain, after expanding (1.3) and (1.4) in a series of powers of $1 - w^2/v_\infty^2$ and using (1.2)

$$\frac{\rho}{\rho_\infty} = 1 + M_\infty^2 \varepsilon A U - \frac{k-1}{2} \varepsilon^2 A^2 U^2 + O(\varepsilon^2 A^2) \tag{1.6}$$

$$\frac{p}{p_\infty} = 1 + k M_\infty^2 \varepsilon A U + 0(\varepsilon^3 A^2) \tag{1.7}$$

Substitution of (1.6) and (1.2) into (1.1) gives, taking (1.5) into consideration:

$$\varepsilon^2 A \left[K \frac{\partial U}{\partial X} + A(k+1) U \frac{\partial U}{\partial X} - \frac{\partial V}{\partial Y}\right] + O(\varepsilon^3 A^2) = 0 \tag{1.8}$$

$$\varepsilon^{5/2} A \left[\frac{\partial U}{\partial Y} + \frac{\partial V}{\partial X}\right] = 0$$

Letting

$$A = \frac{1}{k + 1 + a(1 - M_\infty^2)} \tag{1.9}$$

we obtain the first equation (1.1) satisfied with an accuracy of $\epsilon^3 A^2$ for an arbitrary a, if

$$(K + U) \frac{\partial U}{\partial X} - \frac{\partial V}{\partial Y} = 0 \tag{1.10}$$

The second equation (1.2) is satisfied exactly if

$$\frac{\partial U}{\partial Y} + \frac{\partial V}{\partial X} = 0 \tag{1.11}$$

If we put $U' = K + U$, then equations (1.10) and (1.11) will be the usual equations of transonic flow with respect to U', V.

The parameter a remains undetermined for the time being. If we put $a = 0$, $A = 1/(k + 1)$, then we obtain the approximation due to von Karman [1] and Ovsiannikov [2]. Assuming $a = -k - 1$, $A = 1/(k + 1)M_\infty^2$, we obtain Spreiter's approximation [5].

To determine a, a supplementary condition of best approximation of some gas-dymamic relation is necessary. In studying flows with a passage through the speed of sound, the most important condition concerns the best coincidence of the parabolic lines of the exact systems of equations (1.1) with the approximate system of equations (1.10), (1.11). Thus we try to determine a from the condition of optimum approximation of the relations of the sonic line

$$\left(\frac{w}{v_\infty}\right) = \left(\frac{a_*}{v_\infty}\right)^2 \tag{1.12}$$

Expressing the right-hand side of function (1.12) by means of M_∞ and using (1.2), (1.5), we obtain, taking into account that on the parabolic line

$$2\varepsilon K\left(A - \frac{1}{(k+1)M_\infty^2}\right) + \varepsilon^2 A^2 K^2 + O(\varepsilon^3 A^2) = 0 \tag{1.13}$$

or substituting (1.9) into the right-hand term:

$$\varepsilon^2 A K^2 \left[A - 2\frac{a + k + 1}{(k+1)M_\infty^2}\right] + O(\varepsilon^3 A^2) = 0$$

Employing (1.9) again, we obtain

$$-\varepsilon^2 A^2 K^2 \left[\frac{2a + 2k + 1}{(k+1)M_\infty^2}\right] + O(\varepsilon^3 A^2) = 0$$

It is seen that equation (1.12) is satisfied with an accuracy of $\epsilon^3 A^2$, if $a = -k - ½$ and, therefore,

$$A = \frac{1}{1/2 + (k + 1/2)M_\infty^2} \tag{1.14}$$

As may be seen from (1.13) in Spreiter's approximation ($A = 1/(k + 1)M_\infty$ equation (1.12) is satisfied with an accuracy of $\epsilon^2 A^2$, i.e. lower than if selecting A by formula (1.14).

Let us find the pressure coefficient for the approximation being studied. As is known,

$$\bar{p} = \frac{2}{kM_\infty^2}\left(\frac{p}{p_\infty} - 1\right)$$

Using (1.7) we obtain from the above, with an accuracy of $\epsilon^3 A^2$:

$$\bar{p} = 2\varepsilon A U \tag{1.15}$$

To compare the approximation obtained with the approximation of von Karman and Spreiter, we calculate the pressure coefficient at the sonic point on the profile.

At the sonic point $U = -K$ and, as a consequence, taking into account (1.5), (1.14):

$$\bar{p}_* = 2\,\frac{1 - M_\infty^2}{[{}^1/_2 + (k + {}^1/_2)\,M^2_\infty]^{1/_2}} \tag{1.16}$$

The full line on Fig. 1 shows the exact dependence of p_* on M_∞, the dotted line shows the same function evaluated on the basis of formula (1.16); the dashed line, on the basis of Spreiter's approximation; and the dot-dash line, in accordance with approximation of von Karman.

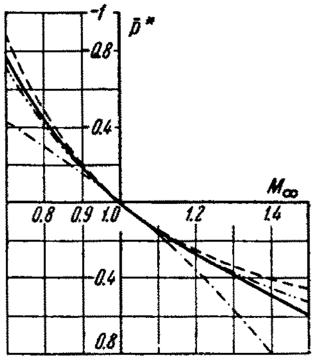

Fig. 1.

2. The relationships on a line of strong discontinuity. Let us clarify the accuracy of the approximations on a line of strong discontinuity under the assumption that the flow beyond it is irrotational and, as a consequence, the density and pressure are calculated by formulas (1.5), (1.7).

The relationships on a line of strong discontinuity are of the form:

$$p - p_1 = \rho_1\,[f'(y)\,v_{y_1} - v_{x_1}]\,(v_x - v_{x_1})$$

$$v_y - v_{y_1} = -f'(y)\,(v_x - v_{x_1})$$

$$\rho\,[f'(y)\,v_y - v_x] = \rho_1\,[f'(y)\,v_{y_1} - v_{x_1}]$$

$$(k+1)\,(p\rho_1 - p_1\rho) = (k-1)\,(p\rho - p_1\rho_1)$$

Here the subscript refers to parameters before the line of strong discontinuity and $x = f(y)$ is the equation of the line of strong discontinuity.

Substitution of (1.6), (1.7), (1.2) into the indicated relations yields

$$O(\varepsilon^3 A^2) = 0, \qquad \varepsilon^{3/2} A\,[(V - V_1) + \Phi'(Y)(U - U_1)] = 0$$

$$\varepsilon^2 A \left[K(U - U_1) + \frac{1}{2}(U^2 - U_1^2) + \Phi'(Y)(V - V_1)\right] + O(\varepsilon^3 A^2) = 0$$

$$O(\varepsilon^3 A^2) = 0$$

where $\Phi'(Y)$ is determined by the relation $f'(y) = \epsilon^{-\frac{1}{2}}\Phi'(Y)$.

It follows that the relationships on the line of strong discontinuity are satisfied with an accuracy of $\epsilon^3 A^2$, for the value of K if

$$V - V_1 + \Phi'(Y)(U - U_1) = 0 \tag{2.1}$$

$$K(U - U_1) + \frac{1}{2}(U^2 - U_1^2) + \Phi'(Y)(V - V_1) = 0 \tag{2.2}$$

If we put $U' = U + K$, then equations (2.1), (2.2) coincide with the usual relations on the line of strong discontinuity in transonic flow [2]. Eliminating $\Phi'(Y)$ from equation (2.1), we obtain the equation of the shock polar line

$$2(V - V_1) + (U + U_1 + 2K)(U - U_1)^2 = 0 \tag{2.3}$$

It should be noted that if (2.3) holds, the exact equation of the shock polar is satisfied with a higher accuracy than the whole system of relations on the line of strong discontinuity. In fact, let us write down the equation of the shock polar, for the sake of simplicity, in the case of an undisturbed flow upstream of the line of strong discontinuity:

$$\left(\frac{v_y}{v_\infty}\right)^2 \left[\frac{2}{k+1} - \frac{v_x}{v_\infty} + \frac{a_*^2}{v_\infty^2}\right] = \left(1 - \frac{v_x}{v_\infty}\right)^2 \left[\frac{v_x}{v_\infty} - \frac{a_*^2}{v_\infty^2}\right] \tag{2.4}$$

Substitution (1.2) yields

$$\varepsilon^2 A^2 \left[\frac{2V^2 + 2KU^2}{(k+1)M_\infty^2} + AU^3\right] + O(\varepsilon^4 A^3) = 0$$

Substituting A by formula (1.9):

$$\frac{\varepsilon^3 A^2}{(k+1)M_\infty^2}[2V^2 + (U + 2K)U^2] - \varepsilon^4 A^3 K \frac{a + k + 1}{(k+1)M_\infty^2} + O(\varepsilon^4 A^3) = 0$$

Considering K to be of the order of unity, we see that for an arbitrary selection of a, equation (2.4) is satisfied with an accuracy of $\epsilon^4 A^3$, if (2.3) is valid, where $U_1 = 0$ is assumed.

If the limitations on K are not imposed, then the same degree of accuracy is obtained if $a = -k - 1$ is selected, that is, it will be the same as in Spreiter's approximation [5]. Since ϵ is connected with the thickness of the profile, it indicates that in Spreiter's approximation the transonic shock polar is very close to the exact polar for arbitrary numbers M_∞ in the neighborhood of a straight jump and weak oblique jumps. It should be noted that in selecting A in accordance with (1.14), the transonic polar coincides best with the exact one in the vicinity of the sonic line.

3. Problem of flow past a profile. Let us consider the boundary conditions of flow past a profile. It is easy to see that the conditions at infinity take the form of:

$$X = \infty, \qquad Y = \infty, \qquad U = V = 0 \tag{3.1}$$

and the conditions on the body

$$v_y = v_x \tau (\bar{\theta} + \bar{\alpha}) \tag{3.2}$$

where τ is the relative thickness of the body, $\bar{\theta} = \theta/\tau$, $\bar{\alpha} = \alpha/\tau$, θ is the angle of inclination of the tangent to the profile with respect to the chord, and α is the angle of attack.

Substitution of (1.2) into the boundary conditions (3.2) yields $\epsilon^{3/2}AV = (1 - \epsilon AU)\tau \ (\bar{\theta} + \bar{\alpha})$. Putting

$$\varepsilon = (\tau/A)^{2/3} \tag{3.3}$$

we obtain the result that (3.2) is satisfied with an accuracy of $\epsilon^{5/2}A^2$, if

$$V = \bar{\theta} + \bar{\alpha} \tag{3.4}$$

Taking into account that equations (1.10), (1.11), relations (2.1), (2.2) and boundary conditions (3.1), (3.4) coincide in the variables $U' = U + K$, V, X, Y with the relations in von Karman's approximation [2], the results of the solution of the problem of flow past a body may be different in various approximations depending upon the choice of A. The solution, as seen from the relations enumerated above, depends on the transonic similarity rule K and $\bar{\alpha} = \alpha/\tau$.

The nearsonic similarity rule, taking (3.3) into consideration, is of the form

$$K = \frac{1 - M_\infty^2}{(\tau/A)^{2/3}} \tag{3.5}$$

and, in particular, for transonic flows

$$K = \frac{1 - M_\infty^2}{\tau^{2/3}\left[{}^1/_2 + (k + {}^1/_2) M_\infty^2\right]^{2/3}}$$

The aerodynamic coefficients are calculated by formulas

$$C_x = \tau^{5/3} A^{1/3} \bar{C}_x, \qquad C_y = \tau^{2/3} A^{1/3} \bar{C}_y, \qquad C_m = \tau^{2/3} A^{1/3} \bar{C}_m \tag{3.6}$$

where $\bar{C}_x$, $\bar{C}_y$, $\bar{C}_m$ are certain integrals along the contour of the profile, and depend on K, $\bar{\alpha}$ and the shape of the profile.

To compare the approximation corresponding to the selection of A by formula (1.14), with the approximation of von Kármán, Spreiter and the exact theory, Fig. 2 and 3 illustrate the dependence between the semi-vertical wedge angle θ_0 and the number M_∞, corresponding to cases of attachment of the shock wave and the passage of flow into a purely supersonic regime.

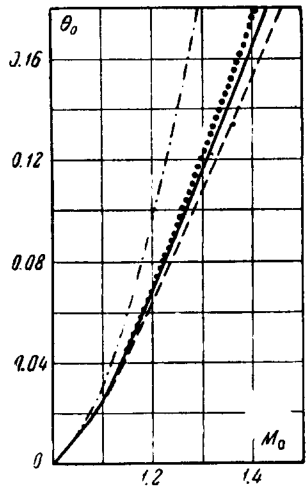

Fig. 2.

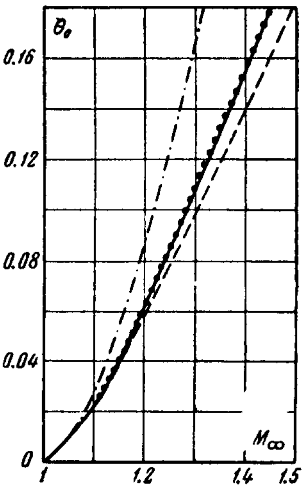

Fig. 3.

The full line indicates the exact theory, the dotted line indicates the approximation given here, the dashed line corresponds to Spreiter's approximation, and the dot-dash line, to von Kármán's approximation.

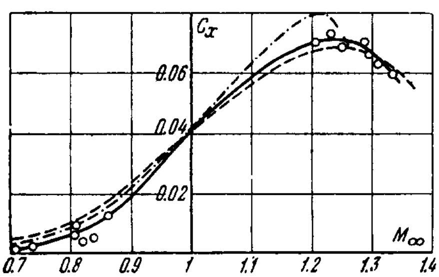

Fig. 4.

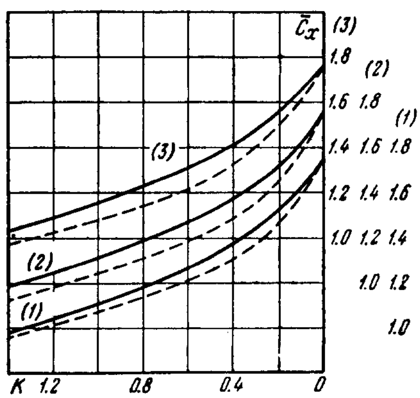

Fig. 5.

Fig. 4 contains the comparison of results of analysis of pressure resistance along the leading edge of the wedge-shaped profile [3,4] on the basis of the approximation given here (full line), Spreiter's approximations (dashed line), von Kármán's approximation (dot-dash line) with experiments [6] (circles) for a wedge angle of $\theta_0 = 7.5^\circ$. Vertical lines indicate the instance of passage of the flow regime into a purely supersonic form.

In conclusion, we remark that in the absence of shock waves (for example, in the pre-critical flow) it is possible not to introduce any limitations on M_∞, assuming $1 - M_\infty^2$ to be sufficiently large as compared to the quantity ϵ. In this case (1.6) takes the form:

$$\rho/\rho_\infty = 1 + M_\infty^2 \varepsilon AU - \frac{M_\infty^2 [(k-2) M_\infty^2 + 1]}{2} \varepsilon^2 A^2 U^2 + O(\varepsilon^3 A^2)$$

and equations (1.1) are satisfied with an accuracy of $\epsilon^2 A^2$, if (1.10), (1.11) are valid and

$$A = 1/M_\infty^2 [(k-2) M_\infty^2 + 3] \tag{3.7}$$

Fig. 5 represents the function $C_x = f(K)$, where C_x is determined by formula (3.6), for laminar flow past the wedge for pre-critical velocities [7]. A is selected: (1) by formula (3.7), (2) after Spreiter or by formula (1.14), (3) after von Kármán. The full line indicates the case of a wedge with a semi-angle of 5°; dashed lines, with a semi-angle of 2.5°. Since for a laminar flow past the wedge for pre-critical velocities the flow does not contain either the sonic or shock waves, the best approximation is obtained, for different wedge angles, if A is determined by formula (3.7).

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